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Derivation of source parameters from low-frequency spectra

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By recording several components of tilt, strain and acceleration at one location, one can determine the focal mechanism, or moment tensor, of an earthquake. Alternatively, recordings made at several locations can be used. The moment tensor can be decomposed into its isotropic part and its deviatoric part. When the eigenvalues of the deviator are in the sequence $(-1, 0, 1)$ the equivalent double couple can be found.

Our objective is to infer the mechanism and moment of an earthquake from observations of the Earth's normal modes excited by it. We use some classical results from the theory of small oscillations (Rayleigh 1877) that have been adapted to our needs (Gilbert 1970). We consider a point source at $\boldsymbol{r}_0$ and let $\boldsymbol{M}$ be the volume integral of the stress drop. Then $\boldsymbol{M}$ has the dimensions of force × length and is called the *moment* or *moment tensor* by seismologists. If the stress drop occurs suddenly enough we use a step function for the time history of the source. Then, the displacement $\boldsymbol{u}(r, t)$ is given by the expression (Gilbert 1970, equation 23)

$$\boldsymbol{u}(\boldsymbol{r}, t) = \sum_n \boldsymbol{S}_n(\boldsymbol{r})\,(\boldsymbol{M}\colon \boldsymbol{\varepsilon}_n(\boldsymbol{r}_0))\,\frac{1-\cos\omega_n t\exp\{-\omega_n t/2Q_n\}}{\omega_n^2}, \tag{1}$$

where $\boldsymbol{S}_n$ is the normalized eigendisplacement for the nth mode, $\boldsymbol{\varepsilon}_n$ is the strain tensor for $\boldsymbol{S}_n$, ω_n is the eigenfrequency and Q_n is the quality factor. Our problem is to determine $\boldsymbol{M}$ given seismograms of $\boldsymbol{u}$ at several locations $\boldsymbol{r}$ on the surface of the Earth. Since $\boldsymbol{u}$ is a linear function of $\boldsymbol{M}$ for any given Earth model the problem is rather straightforward. We depend on having a credible Earth model so that the calculated values of ω_n in (1) are very close to the observed ones. Then we expect the calculated values of $\boldsymbol{S}_n$ to approximate closely those of the real Earth.

Let the six elements of $\boldsymbol{M}$ be M_k, $k = 1, \ldots, 6$ and, similarly, for $\boldsymbol{\varepsilon}_n$, ϵ_{nk}. Let us order our seismograms into the sequence $u_j(t)$, $j = 1, \ldots, J$, and, similarly, the eigendisplacements S_{nj}. Define

$$a_{jk}(t) = \sum_n S_{nj}\,\epsilon_{nk}\frac{1-\cos\omega_n t\exp\{-\omega_n t/2Q_n\}}{\omega_n^2}. \tag{2}$$

Then we can write (1) in the form

$$u_j(t) = \sum_{k=1}^{6} a_{jk}(t)\,M_k \quad (j = 1, \ldots, J). \tag{3}$$

In (3) $a_{jk}(t)$ are calculated for a credible Earth model whose resolvable features are known. At the present time the greatest uncertainty in calculating $a_{jk}(t)$ arises from our poor knowledge about Q_n. We can remove that uncertainty as follows. Let capital letters U and A denote the Fourier transforms of u and a and write the transform of (3)

$$U_j(\omega) = \sum_{k=1}^{6} A_{jk}(\omega)\,M_k \quad (j = 1, \ldots, J). \tag{4}$$

Since A is composed of normal mode line spectra and since the intensity of each line, its spectrum integrated over ω, is independent of Q, then $\bar{U}_j$,

$$\bar{U}_j = \int_W U_j(\omega)\,\mathrm{d}\omega, \tag{5}$$

is independent of Q. In (5) W is a chosen bandwidth. Let

$$\bar{A}_{jk} = \int_W A_{jk}(\omega)\,\mathrm{d}\omega.$$

Then
$$\bar{U}_j = \sum_{k=1}^{6} \bar{A}_{jk} \boldsymbol{M}_k. \tag{6}$$

By assigning to each seismogram a single complex number, its spectrum integrated over bandwidth, we remove the effect of dissipation. If we have six or more carefully chosen seismograms so that $\bar{A}$ in (6) is of full rank, then we can easily determine M in (6), by using, say, Householder's UR method which gives the least squares solution when (6) is overdetermined.

Let us suppose, then, that we have solved (6) and that $\boldsymbol{M}$ is known. We decompose $\boldsymbol{M}$ into its isotropic part, $\frac{1}{3}\boldsymbol{I}\,\mathrm{tr}\,(\boldsymbol{M})$ and its deviatoric part, $\boldsymbol{D}$

$$\boldsymbol{D} = \boldsymbol{M} - \tfrac{1}{3}\boldsymbol{I}\,\mathrm{tr}\,(\boldsymbol{M}). \tag{7}$$

For explosions $\boldsymbol{D}$ is supposed to be zero and for earthquakes $\boldsymbol{M}$ is supposed to be completely deviatoric. Earthquake sources are supposed to be of two types: (1) the compensated, linear, vector dipole and (2) the double couple or fault plane model. A unit compensated, linear, vector dipole, $\boldsymbol{d}_1$, has the representation

$$\boldsymbol{d}_1 = \tfrac{1}{3}(\hat{\boldsymbol{a}}\hat{\boldsymbol{a}} + \hat{\boldsymbol{b}}\hat{\boldsymbol{b}} - 2\hat{\boldsymbol{c}}\hat{\boldsymbol{c}}), \tag{8}$$

where $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$, $\hat{\boldsymbol{c}}$ are three mutually perpendicular unit vectors. A unit double couple $\boldsymbol{q}_1$ as the representation

$$\boldsymbol{q}_1 = \hat{\boldsymbol{s}}\hat{\boldsymbol{f}} + \hat{\boldsymbol{f}}\hat{\boldsymbol{s}}. \tag{9}$$

In (9) $\hat{\boldsymbol{f}}$ is the unit normal to the fault plane. Conventionally $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{f}} > 0$. In (9) $\hat{\boldsymbol{s}}$ is the unit slip vector. It points in the direction of slippage of the hanging wall of the fault with respect to the foot wall.

It has been shown by Knopoff & Randall (1970, p. 4961) that $\boldsymbol{D}$ cannot be decomposed uniquely into a linear combination of $\boldsymbol{d}_1$ and $\boldsymbol{q}_1$, except when the eigenvalues of $\boldsymbol{D}$ have the pattern $(-2, 1, 1)$. Then $\boldsymbol{D}$ is proportional to $\boldsymbol{d}_1$. However, there are many situations where seismologists believe not only that $\boldsymbol{M} = \boldsymbol{D}$ and that the eigenvalues of $\boldsymbol{D}$ have the pattern $(-1, 0, 1)$, but also that $\boldsymbol{D}$ is proportional to $\boldsymbol{q}_1$. In that case the fault plane parameters can be found.

Let $\boldsymbol{D} = \mathscr{M}\boldsymbol{q}_1$; $\mathscr{M}$ is called the (scalar) moment of the earthquake source. Let the eigenvalues of $\boldsymbol{q}_1$ be in the sequence $(-1, 0, 1)$ and let the associated unit eigenvectors be $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$, $\hat{\boldsymbol{w}}$. Then

$$\boldsymbol{q}_1 = \hat{\boldsymbol{w}}\hat{\boldsymbol{w}} - \hat{\boldsymbol{u}}\hat{\boldsymbol{u}}. \tag{10}$$

Since $\hat{\boldsymbol{w}}$ and $\hat{\boldsymbol{u}}$ are determined from the data we can use them to find $\hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{s}}$. From (9) and (10) we have

$$\boldsymbol{q}_1 = \hat{\boldsymbol{s}}\hat{\boldsymbol{f}} + \hat{\boldsymbol{f}}\hat{\boldsymbol{s}} = \hat{\boldsymbol{w}}\hat{\boldsymbol{w}} - \hat{\boldsymbol{u}}\hat{\boldsymbol{u}}. \tag{11}$$

If we write $\hat{\boldsymbol{w}}\hat{\boldsymbol{w}} - \hat{\boldsymbol{u}}\hat{\boldsymbol{u}}$ in the form

$$\boldsymbol{q}_1 = \hat{\boldsymbol{w}}\hat{\boldsymbol{w}} - \hat{\boldsymbol{u}}\hat{\boldsymbol{u}} = \tfrac{1}{2}[(\hat{\boldsymbol{w}} + \hat{\boldsymbol{u}})\,(\hat{\boldsymbol{w}} - \hat{\boldsymbol{u}}) + (\hat{\boldsymbol{w}} - \hat{\boldsymbol{u}})\,(\hat{\boldsymbol{w}} + \hat{\boldsymbol{u}})], \tag{12}$$

we see that we can choose one of $\hat{\boldsymbol{f}}$, $\hat{\boldsymbol{s}}$ to be $2^{-\frac{1}{2}}(\boldsymbol{w} + \boldsymbol{u})$ and the other to be $2^{\frac{1}{2}}(\hat{\boldsymbol{w}} - \hat{\boldsymbol{u}})$ apart from sign conventions. Since $\hat{\boldsymbol{w}}$ belongs to the positive eigenvalue of $\boldsymbol{q}_1$ it points along the tension axis of $\boldsymbol{q}_1$. Similarly, $\hat{\boldsymbol{u}}$ points along the compression axis. Since $\boldsymbol{q}_1$ has been determined for a stress drop, the tension (compression) axis of q_1 is the compression (tension) axis of the source.

In summary, we have seen how to determine the moment tensor from low-frequency spectra. Our method depends upon having a credible Earth model, but is independent of dissipation. When the eigenvalues of the deviator indicate that the source can be represented by a double couple, it is a simple matter to find the fault normal, slip vector, compression axis and tension axis.

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